STRUCTURE OF A VISCOUS ISOELECTRON THERMAL DISCONTINUITY IN A PLASMA

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The structure of a plane stationary shock wave in a dense high-temperature plasma has several distinguishing features. These features have mainly been described and studied in [1–4] (also see survey [5]). As has been demonstrated in these studies, the structure of a shock wave in a plasma is discontinuous for rather large Mach numbers: within the wave, there is a jump in the ion temperature, density, and velocity of the plasma (the electron temperature remains constant). The importance of radiation in this case reduces to some effect on the conditions for formation of the discontinuity if, of course, the photon mean free path is considerably greater than the particle mean free path. This discontinuity is naturally called a viscous isoelectron thermal discontinuity.

The basic mechanism responsible for compression and heating of the plasma ion component in this discontinuity is plasma viscosity, due almost completely to ion collisions. In this article we consider the structure of a discontinuity with allowance for this basic factor, i.e., the ion viscosity of the plasma. In this manner we can obtain many analytic results and can analyze the problem in general form. There is physical justification for ignoring another dissipative process in the plasma-ion component, i.e., ion heat conduction. Heat conduction (Prandtl number $\Pr \approx 2.75^{-1}$) has a very minor effect on the already diffuse viscosity of the discontinuity region, and qualitatively it has no effect on the characteristic picture. This problem has been considered in [6] for the general case of an ideal gas; in this book, it was noted that because of heat conduction the gas entropy within the discontinuity region is nonmonotonic, reaching a maximum at some intermediate point. In our problem, the total entropy, consisting of the sum of ion and electron entropies, has a maximum within the discontinuity even when ion-heat conduction is not allowed for, because electron entropy clearly decreases in the discontinuity.

Allowing for the small thickness of the region of viscous discontinuity (on the order of the particle mean free path l_i) in comparison with the characteristic scale of the structure of the shock wave in the plasma (on the order of $(M/m_e)^{1/2} l_i$, where M and m_e are respectively, the ion and electron masses [3]), we can ignore the change in electron temperature as well as the energy exchange between ions and electrons (due to collisions) in the problem in question. Thus, the problem of the structure of a viscous isoelectron thermal discontinuity reduces to our finding the stationary solution to a system of three nonlinear differential equations defining the velocity, density, and ion temperature of the plasma. We immediately note that this problem differs considerably from the simpler problem of a viscous discontinuity in an ideal gas [7] because the plasma-electron component makes a significant contribution to the pressure gradient owing to change in density.

The detailed structure of a shock wave has already been essentially determined, even without the above approximations, in articles [8–10] which are devoted to a theory of pinch effect in a plasma and which employ numerical methods to solve a nonstationary system of equations. However, to understand several physical problems (for example, electrostatic polarization of a plasma in a shock wave) it is useful to have clearly defined results from the simplified stationary solution. Without going into a derivation and analysis, reference [10] gives some of the formulas which will be obtained below and which were used in calculations of the pinch effect. No new arguments are given with regard to the applicability of the Navier-Stokes approximation used to the problem of shock-wave structure, at least with regard to a more consistent kinetic description (see, for example, [9,10] where these problems are treated in some detail).

§1. Integration of the equations for the structure of a viscous isoelectron thermal discontinuity. We write the equation of motion for a plasma and the equation for heating of the plasma ion component in the plane stationary case, allowing for only ion viscosity [3,8] from among the dissipative processes, and letting $\gamma = 5/3$:

$$\rho v \frac{dv}{dx} = -\frac{dp}{dx} + \frac{4}{3}\frac{d}{dx}\left(\mu \frac{dv}{dx}\right). \qquad (1.1)$$

$$\frac{3}{2}\frac{k}{M}\rho v \frac{dT}{dx} - \frac{k}{M} T v \frac{d\rho}{dx} = \frac{4}{3}\mu\left(\frac{dv}{dx}\right)^2, \qquad (1.2)$$

Here ρ is the plasma density, v is the plasma velocity, T is the ion temperature, k is the Boltzmann constant, and μ is a nonlinear coefficient of viscosity of the plasma ion component. Below we consider only a completely ionized plasma. Then, according to kinetic theory [11],

$$\mu = B\,(kT)^{5/2} = 0.81 M^{1/2} e^{-4} Z^{-4} L^{-1} (kT)^{5/2}, \qquad (1.3)$$

where e is the charge on an electron, Ze is the charge on plasma ions, and L is the Coulomb logarithm. The plasma pressure p which includes the electron-component pressure for the specified constant electron temperature θ_0 is given by

$$p = kM^{-1}\rho\,(T + Z\theta_0). \qquad (1.4)$$

According to the continuity equation

$$\rho v = \rho_0 v_0 = m, \qquad (1.5)$$

where ρ_0 is the initial density ahead of the discontinuity and v_0 is the shock-wave velocity. For the system of equations (1.1–1.5) we must find a solution bounded at infinity.

We allow for (1.5), and, as we know, Eq. (1.1) has the first integral

$$p + mv = p_0 + mv_0 + {}^4/_3\,\mu dv/dx. \qquad (1.6)$$

We set $kM^{-1}Z\theta_0 = C$, express $kM^{-1}T$ in terms of p/ρ, and substitute into Eq. (1.2), eliminating ρ with the aid of (1.5):

$$\frac{3}{2} v \frac{dp}{dx} + \left(\frac{5}{2} p - \frac{Cm}{v}\right)\frac{dv}{dx} = \frac{4}{3}\mu\left(\frac{dv}{dx}\right)^2. \qquad (1.7)$$

Substituting p from (1.6) and dp/dx from (1.1) into (1.7), we obtain an equation in one unknown function v = v(x):

$$\frac{5}{2}(p_0 + mv_0)\frac{dv}{dx} - 4mv\frac{dv}{dx} - \frac{Cm}{v}\frac{dv}{dx} + + 2\frac{d}{dx}\left(\mu v \frac{dv}{dx}\right) = 0. \qquad (1.8)$$

Upon integration, Eq. (1.8) yields the second integral of the problem:

$$^5/_2 (p_0 + mv_0)(v - v_0) - 2m(v^2 - v_0^2) - \\ - Cm \ln v/v_0 + 2\mu v\, dv/dx = 0. \quad (1.9)$$

If we obtain an expression for μ dv/dx from (1.9) and substitute it into (1.6), the pressure p at any point will be defined in terms of the velocity v at this point:

$$p = -mv + \frac{1}{3}(p_0 + mv_0)\left(5\frac{v_0}{v} - 2\right) + \\ + \frac{4}{3}\frac{m}{v}(v^2 - v_0^2) + \frac{2}{3}\frac{Cm}{v}\ln\frac{v}{v_0}. \quad (1.10)$$

Formula (1.10) can be given in some other form if the pressure ahead of the discontinuity p_0 is expressed in terms of the velocity ahead of the discontinuity v_1. From (1.9) we have

$$p_0 + mv_0 = \frac{4}{5}m(v_1 + v_0) + \frac{2}{5}\frac{Cm}{v_1 - v_0}\ln\frac{v_1}{v_0}, \quad (1.11)$$

for the condition dv/dx = 0.

Then with (1.11) formula (1.10) can be converted to the form

$$p = -mv + \frac{4}{3}\frac{m}{v}(v_1 + v_0)\left(v_0 - \frac{2}{5}v\right) + \\ + \frac{4}{3}\frac{m}{v}(v^2 - v_0^2) + \\ + \frac{2}{3}\frac{m}{v}\left(v_0 - \frac{2}{5}v\right)\frac{C}{v_1 - v_0}\ln\frac{v_1}{v_0} + \frac{2}{3}\frac{Cm}{v}\ln\frac{v}{v_0}. \quad (1.12)$$

The pressure p_1 behind the discontinuity is given by (1.12) if we set v = v_1:

$$p_1 = -mv_1 + \frac{4}{5}m(v_1 + v_0) + \frac{2}{5}\frac{Cm}{v_1 - v_0}\ln\frac{v_1}{v_0}. \quad (1.13)$$

On comparing (1.11) and (1.13), we see that because the Hugoniot condition is satisfied momentum is conserved through the discontinuity.

For future use we introduce the dimensionless quantities

$$\pi = \frac{p}{mv_0}, \quad t = \frac{k}{M}T\frac{1}{v_0^2}, \quad \varphi = \frac{C}{v_0^2} = \frac{k}{M}\frac{Z\theta_0}{v_0^2}, \\ u = \frac{v}{v_0}.$$

According to (1.12), the dimensionless pressure is

$$\pi = -u + \frac{4}{3}\frac{(1 + u_1)(1 - {}^2/_5 u)}{u} + \\ + \frac{4}{3}\frac{u^2 - 1}{u} + \frac{2}{3}\frac{\varphi}{u}\frac{1 - {}^2/_5 u}{u_1 - 1}\ln u_1 + \frac{2}{3}\frac{\varphi}{u}\ln u. \quad (1.14)$$

Similarly, Eqs. (1.11) and (1.13) yield

$$\pi_0 = -1 + \frac{4}{5}(1 + u_1) + \frac{2}{5}\frac{\varphi}{u_1 - 1}\ln u_1, \\ \pi_1 = -u_1 + \frac{4}{5}(1 + u_1) + \frac{2}{5}\frac{\varphi}{u_1 - 1}\ln u_1. \quad (1.15)$$

With (1.14), (1.15), and (1.4), we obtain the following expressions for the dimensionless ion temperature t = πu − φ:

$$t = \frac{1}{3}\left[u^2 + 4u_1 - \frac{8}{5}u(1 + u_1)\right] + \\ + \varphi\left[-1 + \frac{2}{3}\ln u + \frac{2}{3}\frac{\ln u_1}{u_1 - 1}\left(1 - \frac{2}{5}u\right)\right], \quad (1.16)$$

$$t_0 = \frac{1}{5}(4u_1 - 1) + \varphi\left(-1 + \frac{2}{5}\frac{\ln u_1}{u_1 - 1}\right), \quad (1.17)$$

$$t_1 = \frac{1}{5}u_1(4 - u_1) + \varphi\left(-1 + \frac{2}{5}\frac{u_1 \ln u_1}{u_1 - 1}\right). \quad (1.18)$$

To complete the solution to the problem it is still necessary to find the form of the function u(x). This can be done with relationship (1.9) by treating it as a differential equation for the inverse function x(u). Here, of course, we assume that u(x) is a monotone function of the x-coordinate (this will be proven rigorously below). Introducing the unit of length

$$x_0 = \frac{1}{m}BM^{1/2}v_0^5 = \frac{0.81}{m}\frac{M^3 v_0^5}{e^4 Z^4 L} \quad (1.19)$$

and the dimensionless coordinate $\xi = xx_0^{-1}$, we obtain from Eq. (1.9) and (1.3)

$$\frac{d\xi}{du} = \\ = \frac{ut^{5/2}}{(1 - u)(u_1 - u) - {}^1/_2\varphi(u - 1)(u_1 - 1)^{-1}\ln u_1 + {}^1/_2\varphi\ln u}. \quad (1.20)$$

It is necessary to substitute p(u) from (1.16) into (1.20) and integrate it from 1 to u_1. Here ξ varies from $-\infty$ to $+\infty$. Relations (1.16) and (1.20) basically contain a solution to the problem posed. For $\varphi = 0$, these relations reduce to familiar expressions and, if the viscosity coefficient does not depend on temperature, Eq. (1.20) is integrated in elementary functions [7].

§2. Qualitative study. Proof of the existence and monotonic nature of the solution. In dimensionless variables, the problem is completely defined by our specifying two quantities: the dimensionless electron temperature φ and the inverse value of compression in the discontinuity u_1. We can show that these quantities are not completely independent. Restrictions on the change in φ follow from the obvious inequalities

$$t_0 \geqslant 0, \quad t_1 \geqslant t_0, \quad \varphi \geqslant 0. \quad (2.1)$$

The second inequality means that in the discontinuity the ion component is heated. In addition, we note that this will also be a necessary condition for the increase in plasma entropy in the discontinuity. From relation (1.17) and the first and third inequalities in (2.1) we have

$$0 \leqslant \varphi \leqslant \frac{{}^1/_5(4u_1 - 1)}{U(u_1)}, \quad U(u_1) = 1 - \frac{2}{5}\frac{\ln u_1}{u_1 - 1}. \quad (2.2)$$

In deriving (2.2) we took account of the properties of the function $U(u_1)$. In the interval $1/4 \leq u_1 \leq 1$, this function increases monotonically from 0.26 to 0.60. Consequently, the denominator in (2.2) is positive in this interval of change in u_1. We can verify that the second inequality in (2.1) imposes a weaker restriction on the quantity φ, i.e., it is satisfied if inequality (2.2) is satisfied. In fact, the second inequality leads to the condition

$$\varphi \leqslant \frac{1 - u_1^2}{-2\ln u_1}. \quad (2.3)$$

It is possible to prove that

$$\frac{{}^1/_5\,(4u_1-1)}{U\,(u_1)} \leqslant \frac{1-u_1^2}{-2\ln u_1}. \qquad (2.4)$$

Allowing for (2.2), inequality (2.4) can be written as

$$1-u_1^2+2u_1\ln u_1 \geqslant 0. \qquad (2.5)$$

As can easily be seen, this inequality holds for

$$0 \leqslant u_1 \leqslant 1,$$

i. e., even for a wider range of change in u_1.

We note that in principle the conditions $t_0 \geq 0$ and $\varphi \geq 0$ permit solutions in which very high compression is involved, when $U(u_1) < 0$, i.e., for u_1 0. Here φ must clearly be greater than the right-hand side of (2.2). The condition $t_1 \geq t_0$ prohibits this solution, because to prove that this solution is not possible we must prove the reverse inequality to (2.4), i.e., we again arrive at (2.5).

Thus, the region of existence for a solution is

$${}^1/_4 \leqslant u_1 \leqslant 1, \; 0 \leqslant \varphi \leqslant f_1(u_1).$$

The shaded region in Fig. 1 represents the region of existence of a solution. Clearly, we have

$$f_1(u_1) = {}^1/_5\,(4u_1-1)/U\,(u_1). \qquad (2.6)$$

We can show that throughout the region of existence t is a monotonically decreasing function of the variable u. We differentiate (1.16) with respect to u:

$$\frac{dt}{du} = \frac{2}{3}\left[u-\frac{4}{5}(1+u_1)\right]+\frac{2}{3}\varphi\left(\frac{1}{u}-\frac{2}{5}\frac{\ln u_1}{u_1-1}\right). \qquad (2.7)$$

Allowing for the obvious inequalities

$$\frac{1}{u}-\frac{2}{5}\frac{\ln u_1}{u_1-1} \geqslant U\,(u_1) > 0,$$

we can introduce a majorant function, setting $\varphi = \varphi_{max} = f_1(u_1)$:

$$\psi(u) = \frac{2}{3}f_1(u_1)\left(\frac{1}{u}-\frac{2}{5}\frac{\ln u_1}{u_1-1}\right)+ \frac{2}{3}\left[u-\frac{4}{5}(1+u_1)\right] \geqslant \frac{dt}{du}. \qquad (2.8)$$

Consider the values of $\psi(u)$ for extreme values of the argument u. First let u = 1. Then, from (2.6) and (2.8), we obtain

$$\psi(1) = 0 \qquad (2.9)$$

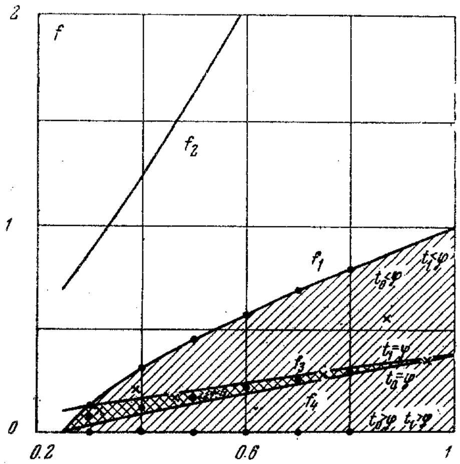

Fig. 1

Moreover, let $u = u_1$; in the same manner, we obtain

$$\psi(u_1) = \frac{2}{15u_1}\,\frac{u_1^2-1-2u_1\ln u_1}{U\,(u_1)} \leqslant 0. \qquad (2.10)$$

The sign of this last expression is obtained from (2.5). Therefore, the derivative dt/du at extreme points of the interval u ($u = u_1$ and u = 1) is not positive and at the point $u = u_1$ it is essentially negative.* Throughout the interval $u_1 < u < 1$ the derivative dt/du is also not positive. This follows from the expression for the derivative of the function $\psi(u)$. From (2.8) we obtain

$$\psi'(u) = -{}^2/_3 u^{-2} f_1(u_1) + {}^2/_3. \qquad (2.11)$$

In accordance with (2.11), the function $\psi(u)$ can decrease for $u > u_1$, beginning with its nonpositive value $\psi = u_1$. Then for $u = u_c = [f_1(u_1)]^{1/2}$ the derivative $\psi'(u)$ vanishes and $\psi'(u) \geq 0$ until u = 1, i.e., $\psi(u)$ increases monotonically from the minimum value of $\psi(u_c)$ to the zero value of $\psi(1) = 0$. If, however, $u_c < u_1$, $\psi(u)$ increases monotonically only from $\psi(u_1)$ to $\psi(1) = 0$.

Thus, we have shown that throughout the region of existence of the solution the ion temperature is a monotonically decreasing function of velocity:

$$dt/du \leqslant 0, \quad u_1 \leqslant u \leqslant 1, \quad {}^1/_4 \leqslant u_1 \leqslant 1, \quad 0 \leqslant \varphi \leqslant f_1(u_1). \qquad (2.12)$$

We demonstrate below that the function $u(\xi)$ in (1.20) also proves to be a monotonically decreasing function throughout the region of existence. To do so, it is necessary and sufficient for us to show that the denominator in the right-hand side of (1.20) is negative everywhere ($u_1 < u < 1$), with the exception of the limit points u = 1 and $u = u_1$, where it vanishes. Thus, we show that the function

$$\chi(u) = (1-u)(u_1-u)-\frac{\varphi}{2}\frac{u-1}{u_1-1}\ln u_1+\frac{\varphi}{2}\ln u < 0, \qquad (2.13)$$

throughout the interval $1 > u > u_1$. The first derivative $\chi'(u)$ is represented by the simple quadratic function

$$\chi'(u) = 2u-(1+u_1)-\frac{\varphi}{2}\left(-\frac{1}{u}+\frac{\ln u_1}{u_1-1}\right). \qquad (2.14)$$

It follows from (2.14) that for u > 0 the equation $\chi'(u)$ has two and only two extrema for any value of φ:

$$u_{\pm} = \frac{1}{4}\left[(1+u_1)+\frac{\varphi}{2}\frac{\ln u_1}{u_1-1}\right] \pm \left\{\frac{1}{16}\left[(1+u_1)+\frac{\varphi}{2}\frac{\ln u_1}{u_1-1}\right]^2-\frac{\varphi}{4}\right\}^{1/2}. \qquad (2.15)$$

We can show that the discriminant of expression (2.15) is positive when $1/4 \leq u_1 \leq 1$, and φ is any positive number; thus, both roots $u_{\pm}$ are real and positive.

From the definition of the function $\chi(u)$ it follows that $\chi(u) \sim (\varphi/2)\ln u \to -\infty$ as $u \to 0$, and that $\chi(u) \sim u^2 \to +\infty$ as $u \to \infty$.

Moreover, the function $\chi(u)$ has at least two of the zeroes indicated above: $u = u_1$ and u = 1. In such a case, the extrema clearly cannot be inflection points; the left extremum ($u = u_-$) must necessarily be a maximum where $\chi(u_-) \geq 0$, and the right extremum ($u = u_+$) must be a minimum where $\chi(u_1) \leq 0$. Moreover, at least one of the extrema must be located between two known zeroes of the function $\chi(u)$. It is now clear that $u_- < u_1$ is a necessary and sufficient condition for inequality (2.13) to be satisfied. However, according to the above the inequality $\chi'(u_1) < 0$ is completely equivalent to this condition. The fact that the conditions $u_- < u_1$ and $\chi'(u_1) < 0$ are

*The equality sign in (2.10) must be omitted since in (2.5) it applies to the case $u_1 = 1$; here $\psi(1) > dt/du$, with the exception of a single unessential point (see Fig. 1) where $\varphi = f_1(1) = 1$ simultaneously with $u_1 = 1$.

equivalent can be proven formally. With (2.14) we write the condition $\chi'(u_1) < 0$ in the explicit form

$$u_1 - 1 + {}^1\!/_2\varphi V(u_1) < 0, \quad V(u_1) = \frac{1}{u_1} - \frac{\ln u_1}{u_1 - 1}. \tag{2.16}$$

It is easy to see that the inequality $V(u_1) \geq 0$ is satisfied over the entire range of change in u_1.* Then inequality (2.16) leads to a restriction on the parameter φ from above:

$$\varphi < 2(1 - u_1)/V(u_1). \tag{2.17}$$

The proof will then be completed by the inequality

$$\varphi_{\max} = f_1(u_1) < f_2(u_1),$$
$$f_2(u_1) = 2(1 - u_1)/V(u_1), \quad {}^1\!/_4 \leqslant u_1 \leqslant 1 \tag{2.18}$$

The functions $f_1(u_1)$ and $f_2(u_1)$ are defined in (2.6) and (2.18).

Inequality (2.18) states that the values of the parameter φ which have physical meaning are smaller than the upper limit set by inequality (2.17). Inequality (2.18) is the easiest to justify, by comparing graphs of the functions $f_1(u_1)$ and $f_2(u_1)$. These graphs have been calculated with sufficient accuracy. This comparison can be made by examining Fig. 1. Essentially, both graphs differ little from the simple linear functions between the limit points $u_1 = 1/4$ and $u_1 = 1$ for which inequality (2.18) is obvious.

Thus, it has been shown that $d\xi/du < 0$ for $1 > u > u_1$, $1/4 \leq u_1 \leq 1$, $0 \leq \varphi \leq f_1(u_1)$. The functions $f_3(u_1)$ and $f_4(u_1)$ are also plotted in Fig. 1. The function $\varphi = f_3(u_1)$ means that the finite ion temperature and the electron temperature $t_1 = \varphi$ are equal, i. e., $t_1 = \varphi$ where t_1 is given by (1.18). Similarly, the function $\varphi = f_4(u_1)$ follows from the fact that the initial ion temperature and the electron temperature $t_0 = \varphi$ are equal, i. e., $t_0 = \varphi$ where t_0 is given by (1.17). Physically, the most interesting region $t_0 < \varphi < t_1$ is located between these two lines and is represented in Fig. 1 by the cross-hatched region. Above this region $t_1 < \varphi$ and $t_0 < \varphi$ and below this region, conversely, $t_1 > \varphi$ and $t_0 > \varphi$.

We summarize the qualitative investigation of the solution. For all values of the parameters u_1 and φ within the limits $1/4 \leq u_1 \leq 1$, $0 \leq \varphi \leq f_1(u_1)$ any isoelectron thermal discontinuity has a very simple continuous structure. The values for the compression and ion temperature increases monotonically in this structure over the entire thickness. We note that the specific relationship between the viscosity coefficient and ion temperature reflects only the effective thickness of the discontinuity (this appears in the numerator of Eq. (1.20)) but has no effect on any of the statements made in this section.

§ 3. Electrostatic polarization of a plasma in any isoelectron thermal discontinuity. From the Boltzmann kinetic equations we can obtain an expression for electric-current density, using the same approximation as for the viscosity tensor. As is known, the basis for this derivation is the Chapman-Enskog method. The following expression can be obtained for the current density in a one-dimensional plane case with no magnetic field and nonzero density and temperature gradients [12]:

$$j = \frac{a(Z)(k\theta)^{3/2}}{m_e^{1/2} e^3 L}\left(\frac{e}{k\theta} E + \frac{1}{\rho}\frac{d\rho}{dx} + \frac{b(Z)}{\theta}\frac{d\theta}{dx}\right). \tag{3.1}$$

Here L is the Coulomb logarithm,† e is the elementary electric charge, $a(Z)$ and $b(Z)$ are functions of Z ($a = 1,2$, $b = 1.7$ for $Z = 1$, and $a = 2.1/Z$, $b = 2.5$ for $Z \gg 1$). In deriving relation (3.1), electron inertia, deviations from quasineutrality, and ion motion were all ignored. Relation (3.1) was also obtained in [13] for more general assumptions as to the state of the plasma; in this article, a system of transport equations for both plasma components was derived directly. Differences in ion and electron temperatures and in ion motion were also considered in [13]. According to [13], Eq. (3.1) contains the electron temperature if $\theta \neq T$. Equation (3.1) also contains the electric-field strength $E = E(x)$. It is quite clear that the stationary electric polarization of the plasma is found from the condition $j = 0$.

The electric field in the steady state can now be found if we use relation (3.1):

$$E_0(x) = -\frac{k\theta}{e\rho}\frac{d\rho}{dx} - \frac{kb(Z)}{e}\frac{d\theta}{dx}. \tag{3.2}$$

The electric charge density is found from the Poisson equation‡:

$$\rho_{e0} = \frac{1}{4\pi}\frac{dE_0}{dx}. \tag{3.3}$$

Later we apply relations (3.2) and (3.3) to the conditions for a viscous isoelectron thermal discontinuity. In the first case, $d\theta/dx = 0$ and $\theta = \theta_0$; in the second case, it is convenient to express all quantities in dimensionless variables. Then, by introducing the quantities ξ, t, and φ, Eqs. (3.2), (1.5), and (1.20) now yield

$$E_0(\xi) = E^\circ \varepsilon(\xi), \quad E^\circ = k\theta_0/ex_0, \tag{3.4}$$

$$\varepsilon(\xi) = (1-u)(u_1-u) - {}^1\!/_2\varphi(u-1)(u_1-1)^{-1}\ln u_1 + {}^1\!/_2\varphi \ln u\left[u^2 t^{5/2}\right]^{-1}. \tag{3.5}$$

By differentiating (3.4), we can obtain from (3.3) an expression for the electric-charge density:

$$\rho_{e0}(\xi) = \rho_e^\circ \sigma(\xi), \quad \rho_e^\circ = E^\circ/4\pi x_0, \quad \sigma(\xi) = d\varepsilon/d\xi. \tag{3.6}$$

The expression for $\sigma(\xi)$ can be written explicitly in terms of u and t if we use (1.16) and (1.20).

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* The inequality $V(u_1) \geq 0$ is satisfied for a wider range of change $0 \leq u_1 \leq 1$, similarly to inequality (2.5). This is a consequence of (2.5) because the elementary transformation $u_1 = x^2$ reduces it to the form $1 - x^2 + 2x^2 \ln x \geq 0$ which clearly exists when (2.5) is satisfied.

† Under ordinary gas-discharge conditions, $L \approx 20$, because in this case L is defined in [12-13] as being twice the value of the Coulomb logarithm (L = 2 Gk35).

‡ As we know, in deriving Eq. (3.1) for j we can ignore the space charge Gk41 ρ_e as long as it is sufficiently small. In this case, its value should be determined from (3.3) [13].

With relations (3.4–3.6) we verify that the electrostatic energy can be ignored in comparison with the thermal energy of the plasma (this fact is made use of in deriving the basic equations)* and that there is justification for assuming the plasma to be quasi-neutral. It is easy to see that the unit of length x_0, defined by relation (1.19), is equal to the effective mean free path of ions with the dimensionless temperature t on the order of unity ($x_0 = 0.81 M^2 \nu_0^4 / n_0 e^4 Z^4 L$) and with the initial plasma density. The relative deviation from quasi-neutrality is obtained from (3.6):

$$\frac{\Delta\rho}{\rho} = \frac{M\rho_e^\circ}{eZ\rho_0}\sigma(\xi)u(\xi) = (Z+1)\frac{D^2}{x_0^2}\sigma(\xi)u(\xi), \qquad (3.7)$$

where we introduce the Debye radius

$$D = \left[\frac{k\theta_0 M}{4\pi e^2 \rho_0 Z(Z+1)}\right]^{1/2}. \qquad (3.8)$$

If, in addition, we introduce the average distance between plasma particles

$$d = [(Z+1)\rho_0 M^{-1}]^{-1/3}, \qquad (3.9)$$

condition (3.7) can be represented in a somewhat different manner by eliminating x_0:

$$\frac{\Delta\rho}{\rho} = \frac{1}{(0.81)^2}\frac{L^2}{(4\pi)^4}\frac{\varphi^4}{Z+1}\left(\frac{d}{D}\right)^6 \sigma(\xi)u(\xi). \qquad (3.10)$$

Throughout the region of existence of the solution $\varphi \leq 1$. Therefore, in (3.10) there is a small factor in front of the dimensionless functions, which in order of magnitude is not greater than $(d/D)^6$. The characteristic ratio d/D appears in uniform-plasma theory and its magnitude is smaller the closer the plasma is to an ideal gas, i.e., the lower the electrostatic energy of the microfields is in comparison with the thermal energy. It is natural to expect the ratio of the electrostatic energy to the thermal energy of the plasma electron component to coincide in order of magnitude with estimates (3.7) or (3.10). It is easy to verify this directly with relations (3.4) and (3.8):

$$\frac{W_e}{W_T} = \frac{E_0^2(\xi)}{8\pi}\left(\frac{3}{2}\frac{k\theta_0 Z\rho}{M}\right)^{-1} = \frac{Z+1}{3}\frac{D^2}{x_0^2}\varepsilon^2(\xi)u(\xi), \qquad (3.11)$$

and by remembering that $\varepsilon_{ef}^2 \sim \sigma_{ef}$, according to (1.20), (3.5), and (3.6).

Under ordinary conditions when Z = 1 and L = 20, the ratio $d/D \approx 10^{-1}$; in any case, it is known to be less than unity for a real plasma.† If we take the ratio $d/D = 10^{-1}$, Eq. (3.10) yields the critical value for the dimensionless product ($\Delta\rho/\rho \approx 1$):

$$|\sigma(\xi)u(\xi)|_* \approx |\sigma(\xi)|_* \approx 10^8 \varphi^{-4}. \qquad (3.12)$$

It is convenient to use this type of relation to justify the initial equations for description of the plasma (as well as the method for calculating the electrostatic field).

For the conditions used, we must have

$$|\sigma(\xi)| \ll |\sigma(\xi)|_* \approx 10^8 \varphi^{-4}. \qquad (3.13)$$

To complete these estimates, we show that the steadiness condition j = 0 is also satisfied when the criteria derived above are realized. By virtue of the equation for continuity of the electric charge, the electric-current density $j_k = v_0 \rho e_0(\xi)$. We take the ratio of this current to the first term in expression (3.1), using (1.19), (3.4), and (3.6):

$$j_k \cdot \left[\frac{a(Z)(k\theta_0)^{3/2}}{m_e^{1/2} e^2 L} E_0(\xi)\right]^{-1} = \qquad (3.14)$$

$$= \left(\frac{m_e}{M}\right)^{1/2} \frac{L^2 \varphi^{3/2}}{0.81(4\pi)^4 Z^{1/2}(Z+1)a(Z)}\left(\frac{d}{D}\right)^6 \frac{\sigma(\xi)}{\varepsilon(\xi)}.$$

From (3.14) it is clear that in general this ratio is also small if d/D is small.

Thus, in a viscous discontinuity, we can justify ignoring plasma polarization in hydrodynamic equations (1.1–1.4). Although polarization has almost no effect on the hydrodynamic and thermodynamic quantities, it is of considerably physical interest in itself. For example, we can consider particle acceleration within the discontinuity. To do this, we estimate the potential difference ΔW at the viscous discontinuity. According to (3.2) and the isothermicity condition

$$\Delta W = -\int_{-\infty}^{+\infty} E_0 dx = \frac{k\theta_0}{e}\ln\frac{\rho_1}{\rho_0} = -\frac{k\theta_0}{e}\ln u_1, \qquad (3.15)$$

the potential difference is basically determined by the electron temperature. It is clear from (3.15) that if a charged particle overcomes this potential difference its energy changes by a value equal to the average electron thermal energy, if $Z \sim 1$. This energy increment will occur over a length on the order of the mean free path of the plasma ions. We can now conclude that plasma particles cannot be effectively accelerated under such conditions, although for a multicharge plasma this process is more significant. Of course, in addition to the electrostatic field, it is also of interest to study the possible electromagnetic radiation of a shock wave in a plasma. However, this effect depends not only on the structure of the viscous discontinuity. It is also governed by the entire configuration of the shock wave in space and by the properties of the environment. We can only indicate here the highest frequency (in order of magnitude) in the signal spectrum $\nu \sim v_0/\xi_{ef} x_0$ where ξ_{ef} is the effective dimensionless width of the discontinuity.

§ 4. Physical results from calculating the structure of a discontinuity. Numerical integration of Eq. (1.20) together with (1.16) and the corresponding boundary

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*Some of these estimates are given in [6], but there they are based on qualitative considerations. The estimates given in our article follow from the general expressions (3.1–3.3) and make allowance for energy relations.

† For example, when $\theta_0 \approx 10^6$ °K, n_0 $6 \cdot 10^{16}$ cm^{-3} (p_0 2 mm Hg) we have $D \approx 2.0 \cdot 10^{-5}$ cm, $d = 2.0 \cdot 10^{-6}$ cm, or when $\theta_0 \approx 10^7$ °K $n_0 \approx 6 \cdot 10^{18}$ cm^{-3}, $D = 6.3 \cdot 10^{-6}$ cm, $d \approx 4.4 \cdot 10^{-7}$. Such conditions are typical of pinch effect in the dense plasma described in [15].

conditions makes it possible, in principle, to find all quantities describing the structure of a discontinuity for any pair of values of the parameters u_1 and φ. The functions $t(\xi)$ and $u(\xi)$ computed in this manner can be lumped with the functions $\pi(\xi)$, $\varepsilon(\xi)$, and $\sigma(\xi)$ if we use (1.14), (3.5), and (3.6), respectively. The table gives certain characteristic quantities, i. e. , the effective width of the discontinuity ξ_{ef},* and the initial and final ion temperatures t_0 and t_1. For each value of the dimensionless velocity behind the discontinuity u_1, three characteristic values are chosen for the dimensionless electron temperature φ. Here the extreme values of φ will be the maximum ones possible (as is clear from Fig. 1 where all variants in the table are denoted by circles). It immediately follows that all variants with $\varphi = 0$ have no immediate physical meaning since in this case the assumption that the discontinuity be isoelectron thermal is not justified. From the relationship between the ion and electron thermal conductivity coefficients,

$$\frac{\varkappa_i}{\varkappa_e} \approx Z^{-3}\left(\frac{m_e}{M}\right)^{1/2}\left(\frac{T}{\theta_0}\right)^{5/2} = \\ = Z^{-1/2}\left(\frac{m_e}{M}\right)^{1/2}\left(\frac{t}{\varphi}\right)^{5/2} \approx \left(\frac{m_e}{M}\right)^{1/2}\left(\frac{t_1}{\varphi}\right)^{5/2}, \qquad (4.1)$$

it follows that to justify the assumption that the discontinuity be isoelectron thermal we must have

$$\varphi > (m_e/M)^{1/5}\, t_1. \qquad (4.2)$$

However, it is easy to see that condition (4.2) will be necessary but not sufficient since the electron temperature is not equalized instantaneously. For this reason, lines in the table with $\varphi = 0$ only serve to estimate the effect of the plasma electron component on the structure of the viscous discontinuity. With this in mind, we can compare lines with $\varphi = 0$ with the average lines for which $t_0 < \varphi < t_1$. When compression is constant and the pressure of the electron component of the plasma is on the order of the ion-component pressure, there is a decrease in the width of the discontinuity by a factor of 2 or 3. Clearly, the main reason for the decrease in ξ_{ef} is the decrease in the finite ion temperature t_1. According to (1.20), $\xi_{ef} \sim t_1^{5/2}$ (more exactly, if we shift from the case $\varphi = 0$ to the case $\varphi \sim t_1$) there is a small counterbalancing effect due to the decrease in the denominator on the right-hand side of (1.20). Incidentally, with (1.17) and (1.18) it is easy to show that throughout the region of existence of the solution (Fig. 1),

$$\partial t_1/\partial\varphi < 0, \;\; \partial t_0/\partial\varphi < 0, \;\; \partial t_1/\partial u_1 > 0. \qquad (4.3)$$

The lines of the table with largest φ also do not have any physical meaning, but for a completely different reason than the lines with $\varphi = 0$. In these variants, because of the extremely small values of ξ_{ef}, electrostatic fields and space charges are obtained which are much greater than their critical values; these can be estimated, for example, with (3.12) and (3.13). All these variants correspond to a zero initial ion temperature $t_0 = 0$. This situation does not occur for a stationary shock-wave structure in a plasma, because owing to energy exchange by collision and the minor amount of plasma compression ahead of the viscous discontinuity the ion temperature becomes nonzero. The discontinuity considered above also cannot occur in the stationary case when $\varphi = 0$. As shown in [4], for Z = 1 and A = 1 (hydrogen plasma) we have $t_0 < \varphi < t_1$. If, however, Z = 3 and A = 7 (lithium plasma), the values of φ are somewhat larger so that they fall within the limits $t_0 < \varphi/3 < t_1$. The crosses in Fig. 1 represent those parameters of viscous discontinuities obtained for stationary shock waves in [4]. The two crosses above the interval $t_0 < \varphi < t_1$ apply to the case Z = 3 and A = 7. For still larger values of Z, the values of φ will be even higher. In the case of an unlimitedly strong stationary shock wave the parameter $u_1 = 0.320$ for Z = 1 and A = 1, while $u_1 = 0.385$ for Z = 3 and A = 7 (see Fig. 1). Therefore, the last line in the table with $u_1 = 0.25$ also has no meaning.

Thus, remembering that our primary concern is with a stationary shock wave in a plasma, we turn our attention to the data in the table for average values of φ.

It is of interest to compare the different regions in which the stationary shock wave occurs. We can limit ourselves to the case of a hydrogen plasma where Z = 1 and A = 1, because here the width of the viscous discontinuity is relatively at a maximum owing to the various dependences of the electron and ion mean free paths on the ion charge Z. For two typical cases we give the effective widths of the stationary shock-wave structure before the discontinuity $\Delta\xi_1$ and after the discontinuity $\Delta\xi_2$, as well as the effective widths of the discontinuity itself (ξ_{ef}), the values of the total reverse compression u_1^* (denoted in [4] as u_2), and the values of the reverse compression in the discontinuity u_1:

u_1	$\Delta\xi_1$	$\Delta\xi_2$	ξ_{ef}	u_1^*
0.75	2.9	1.1	1.0	0.58
0.32	0.11	0.048	0.022	0.25

The quantities $\Delta\xi_1$ and $\Delta\xi_2$ are taken from [4] but are given in terms of the unit of length in this article (1.19). The values of ξ_{ef} are obtained by interpolation between the lines of the table closest to u_1 (we can ignore the difference in the values of the parameter φ). As is clear from these data, for a weak shock wave (upper row) $\xi_{ef} \sim \Delta\xi_2$; even in this case, however, the electron temperature varies little over a length $\sim\Delta\xi_2$ (see Fig. 6 in [4]). Thus, we can assume the isoelectron thermal condition is in effect although there is no great difference between ξ_{ef} and $\Delta\xi_2$ (and $\Delta\xi_1$). The data presented are confirmed qualitatively by numerical calculation (see Fig. 10 of [10] and Fig. 1 of [9]).

In addition, consider criterion (3.13), which indicates a rather low degree of plasma electrostatic polarization. For this purpose, the table gives values

*The effective width of the discontinuity ξ_{ef} is arbitrarily defined here as the distance between two points of the discontinuity structure with $u_0^* = 0.99$ and $u_1^* = u_1 + 0.01$.

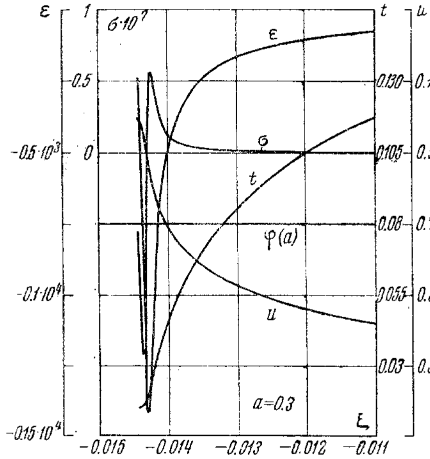

Fig. 2

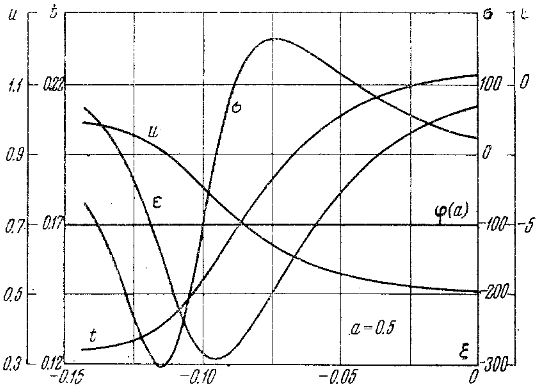

Fig. 3

for $|\sigma|_* = 10^8\varphi^{-4}$ (in accordance with (3.12) and estimated maximum values for $|\sigma|_{max}$ for u_1 and φ. It is clear from the table that criterion (3.13) is satisfied with greatest room to spare even when $u_1 = 0.3$. As follows from the calculations, electrostatic polarization of the discontinuity has a very simple dipole structure with a leading negative-charge maximum roughly

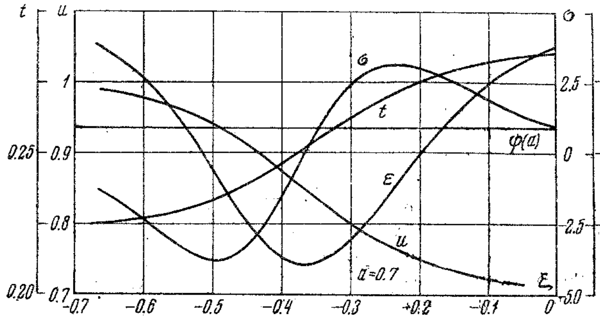

Fig. 4

twice as great as the successive positive-charge maximum ($|\sigma_{min}| \approx 2\sigma_{max}$). For a multicharge plasma, a was shown for Z = 3, the parameter φ lies above the interval $t_0 < \varphi < t_1$. Here ξ_{ef} decreases andl $|\sigma|_{max}$ increases; this increase is roughly proportional to ξ_{ef}^{-2}. In general, however, the critical value of $|\sigma|_*$, also increases since

$$\left(\frac{D}{d}\right)^6 \sim \frac{\theta_0^3}{\rho_0}\frac{1}{Z^3(Z+1)}$$

and the temperature θ_0 for a given density ρ_0 must be considerably greater for a multicharge plasma. A more detailed analysis performed for the specific case of lithium plasma shows that the relation between $|\sigma|_{max}$ and $|\sigma|_*$ remains about the same as for a hydrogen plasma.

Nevertheless, if criterion (3.13) is violated for any reason, the preceding method of considering the structure of a viscous isoelectron thermal discontinuity is no longer applicable. It would then be necessary to solve a considerably more complex problem with allowance in the initial equations for the difference in electron and ion velocities and densities. In other words, it would be necessary to allow for the countering effect of the electric field on plasma motion. This was done in [16] for some particular cases. We can

u_1	φ	ξ_{ef}	t_0	t_1	$\|\sigma\|_*$	$\|\sigma\|_{max}$
0.8	0.795	$4.2\cdot10^{-7}$	0	$1.06\cdot10^{-3}$		
	0.295	1.43	0.278	0.320	$1.32\cdot10^{10}$	0.4
	0	4.08	0.443	0.508		
0.7	0.686	$8.2\cdot10^{-6}$	0	$4.02\cdot10^{-3}$		
	0.260	0.672	0.225	0.286	$2.19\cdot10^{10}$	4
	0	1.99	0.362	0.458		
0.6	0.572	$7.0\cdot10^{-5}$	0	$1.09\cdot10^{-2}$		
	0.220	0.316	0.173	0.253	$4.26\cdot10^{10}$	30
	0	0.944	0.282	0.403		
0.5	0.449	$3.96\cdot10^{-4}$	0	$2.52\cdot10^{-2}$		
	0.170	0.143	0.125	0.224	$1.19\cdot10^{11}$	$3\cdot10^{2}$
	0	0.402	0.201	0.345		
0.4	0.308	$1.78\cdot10^{-3}$	0	$5.4\cdot10^{-2}$		
	0.120	$5.46\cdot10^{-2}$	$7.4\cdot10^{-2}$	0.194	$4.85\cdot10^{11}$	$6\cdot10^{5}$
	0	0.140	0.121	0.283		
0.3	0.128	$7.17\cdot10^{-3}$	0	0.102		
	0.080	$1.44\cdot10^{-2}$	$1.5\cdot10^{-2}$	0.155	$2.44\cdot10^{12}$	$1.4\cdot10^{7}$
	0	$3.38\cdot10^{-2}$	$4\cdot10^{-2}$	0.217		
0.25	0	$1.45\cdot10^{-2}$	0	0.182		

expect (3.13) to be violated, for example, in the complex structure of a nonstationary shock wave or in the case of a very dense plasma in which the ratio d/D increases and approaches unity.

More detailed information as to the structure of a viscous isoelectron thermal discontinuity is given in Figs. 2–4, where profiles of the functions t, u, ε, and σ are constructed as functions of the coordinates ξ. Here the various figures differ in the values for the finite plasma velocity u_1; these values are respectively 0.7, 0.5, and 0.3. Values for the parameter φ are chosen in the same manner as in the table.

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